

A General Criterion for the Dynamical Stability of Anisotropic Newtonian Systems

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The problem of the dynamical stability of anisotropic systems is studied, by proposing a criterion in terms of the adiabatic local index γ . The result has general validity and can be applied to several physical situations. The stability can be achieved either for $\gamma \geq 4/3$ or $\gamma < 4/3$, according to the equation of state of the fluid considered. Configurations that can not exist in the isotropic regime can exist in the anisotropic one. Some applications of the criterion are also included.

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I. INTRODUCTION

The stability of self-gravitating systems is an old but still active topic of research that can be divided in two main branches, thermodynamical and dynamical. The former refers to the description, from a thermodynamical point of view, of the properties of systems kept bound by gravitational forces [1, 2]. The latter constitutes instead the subject of this paper.

The pioneering study about the dynamical stability was proposed by Jeans [3] (see also Ref. [4]), who found a criterion (which extensions in presence of magnetic fields are in Refs. [5, 6]) in terms of a characteristic parameter (the Jeans length λ_J). According to this criterion, there is instability (and consequently the collapse of the structure) if the size of the system under consideration is larger than λ_J .

A second way of studying dynamical stability is represented by the variational principles, subsequently applied to barotropic stars [2]. Nevertheless, the exact approach is based on the solution of the pulsation equation [7], by considering radial perturbations. In addition, from this equation, it is possible to derive a criterion in terms of the adiabatic local index γ (e.g. Refs. [8, 9]), both in Newtonian gravity and in General Relativity.

Anyway several proposals, for what concerns stability criteria, have been advanced. Ipser and Thorne [10] (see also Refs. [11–16]) firstly proposed a criterion for the spherically relativistic star clusters. Moreover, Herrera *et al.* [17] studied the problem of the dynamical instability for systems undergoing a non-adiabatic spherical collapse (the extension of this model to anisotropic systems can be found in Ref. [18]), by providing several expressions of the adiabatic index in relativistic, Newtonian and post-Newtonian limits.

All the works mentioned up to this point considered

the problem in the hypothesis of isotropy. However, a description taking into account anisotropy is necessary, because a large number of astrophysical systems presents some features that can not be explained by means of isotropic models (see Refs. [19, 20] and references therein).

Moved from these reasons, Dorémus *et al.* [21] firstly studied the dynamical stability of this class of objects, by considering stellar systems (see also Ref. [22]). A generalization of Jeans criterion to anisotropic fluids has been proposed by Herrera and Santos [23], by showing as well that the critical mass to start the collapse (Jeans mass M_J , defined as the mass within a sphere of radius equal to λ_J) is smaller than the corresponding one evaluated in the hypothesis of isotropy. Other models, about the stability of anisotropic stars, can be found in Refs. [24–30].

In this work, we aim at finding a stability criterion in terms of the adiabatic local index γ , by following the same approach proposed by Dev and Gleiser [26]. In the next section we derive the pulsation equation and, in Sec. III, the stability criterion. Furthermore, in Sec. IV we provide some applications of the criterion. In Sec. V, finally, we draw some conclusions.

II. DEDUCTION OF THE PULSATION EQUATION FOR ANISOTROPIC SYSTEMS

Let us start by considering the equations governing the gravitational equilibrium (primes indicate the derivative respect to r)

$$\begin{aligned} P'_{rr} &= -\rho\Phi' - \frac{2(P_{rr} - P_t)}{r} = -\rho\Phi' - \frac{2\Pi}{r}, \\ M'_r &= 4\pi\rho r^2. \end{aligned} \quad (1)$$

Φ is the gravitational potential, ρ the matter density and M_r the mass confined within the sphere of radius r . In our deduction we follow the formalism outlined in Ref.

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[31]. The dynamics of an unperturbed and static system (in spherical symmetry) is governed by the following equations

$$\frac{\partial \rho}{\partial t} + (\rho v)' = 0, \quad (2)$$

$$\frac{\partial v}{\partial t} = -\frac{P'_{rr}}{\rho} - \Phi' - \frac{2\Pi}{r\rho}, \quad (3)$$

$$\frac{(r^2\Phi')'}{r^2} = 4\pi G\rho \quad (4)$$

i.e. continuity, Euler and Poisson equation, respectively. In order to obtain the equation regulating the stability, we have to calculate the variations of the quantities in Eq.(3). For a generic quantity $Q = Q(\vec{r}, t)$ we define the Eulerian variation as

$$\delta Q = Q(\vec{r}, t) - Q_0(\vec{r}, t). \quad (5)$$

The Lagrangian variation is instead defined by the following relation

$$\Delta Q = Q[\vec{r} + \vec{\xi}(\vec{r}, t), t] - Q_0(\vec{r}, t) = \delta Q + \vec{\xi} \cdot \nabla Q, \quad (6)$$

where $\vec{\xi}$ is the Lagrangian displacement. Now, by considering Euler equation, the Lagrangian variation is

$$\Delta\left(\frac{\partial v}{\partial t}\right) = -\Delta\left(\frac{P'_{rr}}{\rho}\right) - \Delta(\Phi') - \Delta\left(\frac{2\Pi}{r\rho}\right). \quad (7)$$

For the term on the l.h.s. of Eq.(7) we note that

$$\Delta v = \frac{d(r + \xi)}{dt} - \frac{dr}{dt} = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} \quad (8)$$

(the unperturbed system is static). Thanks to the preceding relation we may write $\Delta(\partial v / \partial t) = \partial^2 \xi / \partial t^2$ and $\Delta r = \xi$ (see Ref. [31] for more details). On the other hand, the Lagrangian variation of the matter density can be obtained from the equation of the conservation of mass

$$\Delta \rho = -\rho \nabla \cdot \vec{\xi} = -\frac{\rho}{r^2} (r^2 \xi)'. \quad (9)$$

Let us now consider

$$\begin{aligned} \Delta\left(\frac{P'_{rr}}{\rho}\right) &= \frac{\Delta(P'_{rr})}{\rho} - \frac{P'_{rr}\Delta\rho}{\rho^2} = \\ &= \frac{\Delta(P'_{rr})}{\rho} + \frac{P'_{rr}}{\rho} \frac{(r^2 \xi)'}{r^2}. \end{aligned} \quad (10)$$

By defining γ , the adiabatic local index of the perturbations, as

$$\gamma = \frac{\rho}{P_{rr}} \frac{\Delta P_{rr}}{\Delta \rho} \implies \Delta P_{rr} = -\frac{\gamma P_{rr}}{r^2} (r^2 \xi)', \quad (11)$$

the variation of P'_{rr} is

$$\Delta(P'_{rr}) = (\Delta P_{rr})' - \xi' P'_{rr} = -\left[\frac{\gamma P_{rr}}{r^2} (r^2 \xi)'\right]' - \xi' P'_{rr} \quad (12)$$

and, substituting in Eq.(10), we get

$$\Delta\left(\frac{P'_{rr}}{\rho}\right) = -\frac{1}{\rho} \left[\frac{\gamma P_{rr}}{r^2} (r^2 \xi)'\right]' + \frac{2\xi P'_{rr}}{\rho r}. \quad (13)$$

For what concerns the term $\Delta(\Phi')$, let us consider

$$(\delta \Phi)' = \delta(\Phi') = G \frac{\delta M_r}{r^2} = -4\pi G \rho \xi, \quad (14)$$

because of the relation $\Delta M_r = \delta M_r + 4\pi \rho r^2 \xi = 0$. By using Eqs.(4) and (14) we have

$$\begin{aligned} \Delta(\Phi') &= (\delta \Phi)' + \xi \Phi'' = -4\pi G \rho \xi + \\ &+ \xi \left(\nabla^2 \Phi - \frac{2\Phi'}{r} \right) = -\frac{2\xi \Phi'}{r}, \end{aligned} \quad (15)$$

and, thanks to the equation of hydrostatic equilibrium, we get

$$\Delta(\Phi') = -\frac{2\xi P'_{rr}}{\rho r} + \frac{4\xi \Pi}{\rho r^2}. \quad (16)$$

By considering the last term of Eq.(7)

$$\begin{aligned} \Delta\left(\frac{2\Pi}{r\rho}\right) &= \frac{2\Delta\Pi}{r\rho} - \frac{2\Pi\Delta\rho}{r\rho^2} - \frac{2\Pi\Delta r}{r^2\rho} = \\ &= \frac{2\Delta\Pi}{r\rho} + \frac{2\Pi\xi}{r^2\rho} + \frac{2\Pi\xi'}{r\rho} \end{aligned} \quad (17)$$

we have, finally

$$\begin{aligned} \Delta\left(\frac{\partial v}{\partial t}\right) &= \frac{\partial^2 \xi}{\partial t^2}, \\ \Delta\left(\frac{P'_{rr}}{\rho}\right) &= -\frac{1}{\rho} \left[\frac{\gamma P_{rr}}{r^2} (r^2 \xi)'\right]' + \frac{2\xi P'_{rr}}{\rho r}, \\ \Delta(\Phi') &= \frac{2\xi P'_{rr}}{\rho r} + \frac{4\Pi\xi}{\rho r^2}, \\ \Delta\left(\frac{2\Pi}{r\rho}\right) &= \frac{2\Delta\Pi}{r\rho} + \frac{2\Pi\xi}{r^2\rho} + \frac{2\Pi\xi'}{r\rho}. \end{aligned} \quad (18)$$

Substituting the relations (18) in Eq.(7), we obtain the general pulsation equation for anisotropic systems

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{4\xi P'_{rr}}{r\rho} - \frac{1}{\rho} \left[\frac{\gamma P_{rr}}{r^2} (r^2 \xi)'\right]' + \frac{6\Pi\xi}{r^2\rho} + \frac{2\Pi\xi'}{r\rho} + \frac{2\Delta\Pi}{r\rho} = 0. \quad (19)$$

III. EXPRESSION OF γ_{cr} AND STABILITY CRITERION

In order to obtain the stability criterion, we have to express $\Delta\Pi$ in terms of ξ . There are two methods: the first one is to evaluate $\Delta\Pi = \delta\Pi + \xi\Pi'$, the second one write $\Delta\Pi$ in terms of variables which we know the Lagrangian variation. Since an expression for δP_t is missing, we use the second method. Without losing in generality, we can suppose that $\Pi = \Pi(r, \rho)$: in this case we may write $\Delta\Pi$ as

$$\Delta\Pi = \frac{\partial\Pi}{\partial r}\Delta r + \frac{\partial\Pi}{\partial\rho}\Delta\rho = \frac{\partial\Pi}{\partial r}\xi - \frac{\rho}{r^2}(r^2\xi)'\frac{\partial\Pi}{\partial\rho}. \quad (20)$$

Substituting Eq.(20) in Eq.(19), we get

$$\begin{aligned} \frac{\partial^2\xi}{\partial t^2} + \frac{4\xi P'_{rr}}{r\rho} - \frac{1}{\rho}\left[\frac{\gamma P_{rr}}{r^2}(r^2\xi)'\right]' + \frac{6\Pi\xi}{r^2\rho} + \frac{2\Pi\xi'}{r\rho} + \\ + \frac{2}{r\rho}\left[\frac{\partial\Pi}{\partial r}\xi - \frac{\rho(r^2\xi)'}{r^2}\frac{\partial\Pi}{\partial\rho}\right] = 0. \end{aligned} \quad (21)$$

By assuming $\xi(r, t) = \xi(r)e^{-i\omega t}$ (ω is the pulsation frequency), Eq.(21) becomes

$$\begin{aligned} \omega^2\rho\xi = \frac{4\xi P'_{rr}}{r} - \left[\frac{\gamma P_{rr}}{r^2}(r^2\xi)'\right]' + \frac{6\Pi\xi}{r^2} + \frac{2\Pi\xi'}{r} + \\ + \frac{2\xi}{r}\frac{\partial\Pi}{\partial r} - \frac{2\rho(r^2\xi)'}{r^3}\frac{\partial\Pi}{\partial\rho}, \end{aligned} \quad (22)$$

with the boundary conditions $\xi(0) = 0$ and $\Delta P_{rr}(R) = 0$. It is easy to show that, for $P_{rr} = P_t = P$, Eq.(22) recovers the pulsation equation for isotropic systems [32]. The pulsation equation (22) represents a Sturm - Liouville eigenvalue problem. It is well known from the theory of these equations (see e.g. Ref. [33]) that there is an infinite number of real eigenvalues ω_n^2 , where n is an integer. The corresponding eigenfunctions ξ_n form an orthonormal basis with the weighting function ρr^2 . The system is stable if $\omega_0^2 > 0$ (fundamental mode): as a consequence, all the other modes are stable. On the contrary, if $\omega_0^2 < 0$, the system is dinamically unstable. When $\omega_0^2 = 0$, the system is marginally stable. Now, multiplying both members of the foregoing expression by $\xi r^2 dr$ and integrating from 0 to R , we have

$$\begin{aligned} \omega^2 \int_0^R \rho \xi^2 r^2 dr = 4 \int_0^R P'_{rr} \xi^2 r dr + 6 \int_0^R \Pi \xi^2 dr + \\ - \int_0^R \left[\frac{\gamma P_{rr}}{r^2}(r^2\xi)'\right]' \xi r^2 dr + 2 \int_0^R \Pi \xi' \xi r dr + \\ + 2 \int_0^R \frac{\partial\Pi}{\partial r} \xi^2 r dr - 2 \int_0^R \frac{\rho(r^2\xi)'}{r} \frac{\partial\Pi}{\partial\rho} \xi dr. \end{aligned} \quad (23)$$

Integrating by parts the first two terms on the r.h.s. of Eq.(23), we get

$$\begin{aligned} \omega^2 \int_0^R \rho \xi^2 r^2 dr = 2 \int_0^R \Pi \xi' \xi r dr + 6 \int_0^R \Pi \xi^2 dr + \\ + \int_0^R \frac{\gamma P_{rr}[(r^2\xi)']^2}{r^2} dr - 4 \int_0^R P_{rr}(\xi^2 r)' dr + \\ + 2 \int_0^R \frac{\partial\Pi}{\partial r} \xi^2 r dr - 2 \int_0^R \frac{\rho(r^2\xi)'}{r} \frac{\partial\Pi}{\partial\rho} \xi dr. \end{aligned} \quad (24)$$

At this point, to obtain the criterion, we have to apply the stability condition above explained: this implies that

$$\begin{aligned} \int_0^R \frac{\gamma P_{rr}}{r^2} [(r^2\xi)']^2 dr - 4 \int_0^R P_{rr}(\xi^2 r)' dr + \\ + 6 \int_0^R \Pi \xi^2 dr + 2 \int_0^R \Pi \xi' \xi r dr + \\ + 2 \int_0^R \frac{\partial\Pi}{\partial r} \xi^2 r dr - 2 \int_0^R \frac{\rho(r^2\xi)'}{r} \frac{\partial\Pi}{\partial\rho} \xi dr \geq 0. \end{aligned} \quad (25)$$

By introducing the homologous displacement [7] and using the definition of Π , we have

$$\begin{aligned} 9 \int_0^R \gamma P_{rr} r^2 dr \geq 4 \int_0^R P_{rr} r^2 dr + 8 \int_0^R P_t r^2 dr + \\ - 2 \int_0^R \frac{\partial\Pi}{\partial r} r^3 dr + 6 \int_0^R \frac{\partial\Pi}{\partial\rho} \rho r^2 dr. \end{aligned} \quad (26)$$

Dividing both members by $9 \int_0^R P_{rr} r^2 dr$, simplifying and defining

$$<\gamma> = \frac{\int_0^R \gamma P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr}, \quad (27)$$

$$\begin{aligned} \gamma_{cr} = \frac{4}{9} + \frac{8}{9} \frac{\int_0^R P_t r^2 dr}{\int_0^R P_{rr} r^2 dr} - \frac{2}{9} \frac{\int_0^R (\partial_r \Pi) r^3 dr}{\int_0^R P_{rr} r^2 dr} + \\ + \frac{2}{3} \frac{\int_0^R (\partial_\rho \Pi) \rho r^2 dr}{\int_0^R P_{rr} r^2 dr}, \end{aligned} \quad (28)$$

we get

$$<\gamma> \geq \gamma_{cr}. \quad (29)$$

Moreover, for $P_{rr} = P_t = P$, Eq.(28) yields $\gamma_{cr} = 4/3$.

IV. APPLICATIONS

A. Systems of Fermions

In this section, we analyze the dynamical stability of fermionic systems studied in Ref. [19] (hereafter Paper

I). In this model, P_{rr} and P_t do not explicitly depend on the matter density (see Paper I for more details). Consequently, Eq.(28) becomes

$$\gamma_{cr} = \frac{4}{9} + \frac{8}{9} \frac{\int_0^R P_t r^2 dr}{\int_0^R P_{rr} r^2 dr} - \frac{2}{9} \frac{\int_0^R \Pi' r^3 dr}{\int_0^R P_{rr} r^2 dr}. \quad (30)$$

Integrating by parts the last term on the r.h.s. of Eq.(30), we have

$$\int_0^R \Pi' r^3 dr = -3 \int_0^R (P_{rr} - P_t) r^2 dr. \quad (31)$$

because, according to the definitions given in Paper I, we have $P_{rr} \rightarrow 0$ and $P_t \rightarrow 0$ in the limit $r \rightarrow R$. Substituting now in Eq.(30) we get

$$\gamma_{cr} = \frac{10}{9} + \frac{2}{9} \frac{\int_0^R P_t r^2 dr}{\int_0^R P_{rr} r^2 dr}. \quad (32)$$

In Paper I we have further defined the anisotropy level parameter

$$\eta = \frac{P_{rr}}{P_t} \quad \text{with} \quad \frac{1}{2} \leq \eta \leq 1. \quad (33)$$

The lower limit has been obtained in the full anisotropic regime whereas the upper one in the isotropic limit. Thanks to the foregoing expression, Eq.(32) becomes

$$\gamma_{cr} = \frac{2}{9} \left(5 + \frac{1}{\eta} \right) \quad \text{with} \quad \frac{4}{3} \leq \gamma_{cr} \leq \frac{14}{9}. \quad (34)$$

It should be stressed that Eq.(34) does not depend on the statistics, by implying that it is possible to analyze the dynamical stability of systems composed by different particles but described by the same pressure tensor. In Fig. 1 we have represented the behavior of γ_{cr} .

B. Anisotropic Stars: the Model of Heintzmann and Hillebrandt

In this section we refer to the paper of Heintzmann and Hillebrandt ([24], hereafter HH75). The tensor pressure is expressed by

$$P_t = (1 + \beta) P_{rr} \quad \Rightarrow \quad \Pi = -\beta P_{rr}, \quad (35)$$

with $\beta = \beta(r, \rho)$. Let us substitute Eq.(35) in Eq.(28): we get

$$\begin{aligned} \gamma_{cr} = & \frac{4}{3} + \frac{8}{9} \frac{\int_0^R \beta P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr} + \\ & + \frac{2}{9} \frac{\int_0^R \partial_r(\beta P_{rr}) r^3 dr}{\int_0^R P_{rr} r^2 dr} - \frac{2}{3} \frac{\int_0^R \partial_\rho(\beta P_{rr}) \rho r^2 dr}{\int_0^R P_{rr} r^2 dr}. \end{aligned} \quad (36)$$

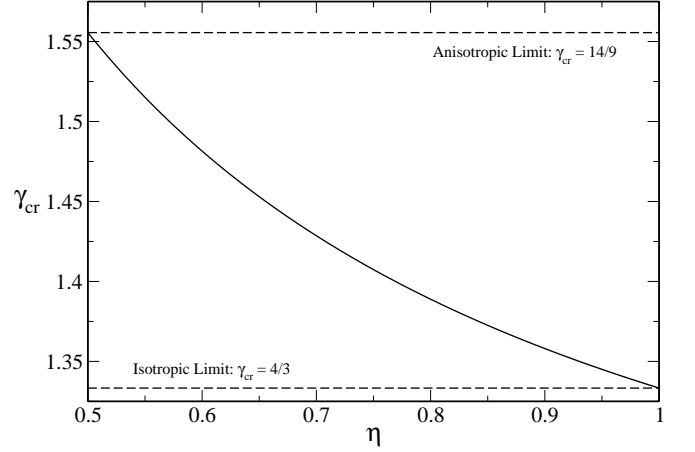


FIG. 1. γ_{cr} as a function of η , according to Eq.(34). The limiting values are also indicated.

The second integral on the r.h.s of Eq.(36) may be written, integrating by parts, as

$$\int_0^R \frac{\partial(\beta P_{rr})}{\partial r} r^3 dr = -3 \int_0^R \beta P_{rr} r^2 dr. \quad (37)$$

Furthermore, we have

$$\frac{\partial(\beta P_{rr})}{\partial \rho} = P_{rr} \frac{\partial \beta}{\partial \rho} + \beta \frac{\partial P_{rr}}{\partial \rho} = P_{rr} \frac{\partial \beta}{\partial \rho} + \beta \frac{\gamma P_{rr}}{\rho}. \quad (38)$$

Substituting Eqs.(37)-(38) in Eq.(39), we have

$$\begin{aligned} \gamma_{cr} = & \frac{4}{3} + \frac{2}{9} \frac{\int_0^R \beta P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr} - \frac{2}{3} \frac{\int_0^R (\partial_\rho \beta) P_{rr} \rho r^2 dr}{\int_0^R P_{rr} r^2 dr} + \\ & - \frac{2}{3} \frac{\int_0^R \beta \gamma P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr}. \end{aligned} \quad (39)$$

Now, by using Eq.(29), we obtain

$$\begin{aligned} \gamma_\star = & \frac{\int_0^R (2\beta + 3) \gamma P_{rr} r^2 dr}{3 \int_0^R P_{rr} r^2 dr} \geq \frac{4}{3} + \frac{2}{9} \frac{\int_0^R \beta P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr} + \\ & - \frac{2}{3} \frac{\int_0^R (\partial_\rho \beta) P_{rr} \rho r^2 dr}{\int_0^R P_{rr} r^2 dr}. \end{aligned} \quad (40)$$

Eq.(40) represents the most general case for systems described by the pressure tensor (35). It is possible to study the stability of this class of systems only having an expression of the function $\beta = \beta(r, \rho)$ (an example is in HH75). Anyway, for $\beta = \text{const}$, we obtain

$$\gamma_\star = \langle \gamma \rangle = \frac{\int_0^R \gamma P_{rr} r^2 dr}{\int_0^R P_{rr} r^2 dr} \geq \frac{2(6 + \beta)}{3(3 + 2\beta)} = \gamma_{cr}, \quad (41)$$

which is shown in Fig. 2.

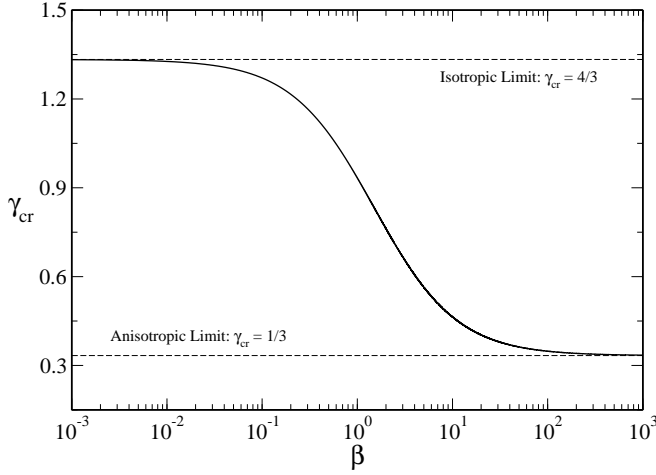


FIG. 2. γ_{cr} as a function of β , according to Eq.(41). The isotropic limit corresponds to $\beta \rightarrow 0$, the anisotropic one to $\beta \rightarrow \infty$.

C. Anisotropic Stars: the Model of Dev and Gleiser

In this section we refer to the paper of Dev and Gleiser [26]. The stability has been studied by considering three different equations of state at constant density $\rho = \rho_0$

$$\begin{aligned} 1) \quad \Pi &= -C\rho_0^2 r^2, \\ 2) \quad \Pi &= -C\rho_0 P_{rr} r^2, \\ 3) \quad \Pi &= -CP_{rr}^2 r^2. \end{aligned} \quad (42)$$

In the previous relations, C represents the strenght of anisotropy and can be both positive and negative. The corresponding expressions of γ_{cr} are

$$\begin{aligned} 1) \quad \gamma_{cr} &= \frac{4}{3} \left(1 + \frac{3}{2} \frac{1}{\frac{2\pi}{3C} - 1} \right), \\ 2) \quad \gamma_{cr} &= \frac{4}{3} + \frac{2C\rho_0}{9} \frac{F_1(r) \int_0^R e^{C\rho_0 r^2} dr}{F_2(r) \int_0^R e^{C\rho_0 r^2} dr}, \\ 3) \quad \gamma_{cr} &= \frac{4}{3} - \frac{8C}{9} \frac{\int_0^R P_{rr}^2 r^4 dr}{\int_0^R P_{rr} r^2 dr}, \end{aligned} \quad (43)$$

where the functions $F_1(r)$ and $F_2(r)$ are defined by

$$\begin{aligned} F_1(r) &= \frac{R^5}{5} - \frac{R^3}{2C\rho_0} + \frac{3R}{4C^2\rho_0^2} - \frac{3e^{-C\rho_0 R^2}}{4C^2\rho_0^2}, \\ F_2(r) &= \frac{R^3}{3} - \frac{R}{2C\rho_0} + \frac{e^{-C\rho_0 R^2}}{2C\rho_0}. \end{aligned} \quad (44)$$

The reader can find the expression of P_{rr} for the third equation of state in Ref. [26]. In Fig. 3 we have represented γ_{cr} for the first equation of state.

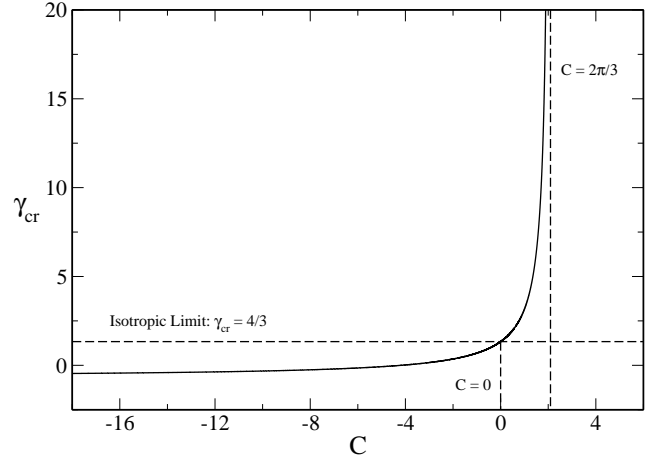


FIG. 3. γ_{cr} as a function of C , according to the first of Eqs.(43). The critical points for which $\gamma_{cr} = 4/3$ and $\gamma_{cr} \rightarrow \infty$ are also indicated.

D. Generalized Polytropes

For the deduction of γ_{cr} we refer to the paper of Herrera and Barreto [34], where the general formalism for polytropes in presence of anisotropy has been derived. They considered

$$\Pi = -Cf\rho r^N, \quad (45)$$

where C is a constant (with the same meaning as explained in Sec. IV C), $f = f(r)$ and N are to be specific for each model. Substituting the previous relation in Eq.(28) and rearranging the terms, we have

$$\gamma_{cr} = \frac{4}{3} + \frac{2C}{9} \frac{\int_0^R [(N+1)f + rf'] \rho r^{N+2} dr}{\int_0^R P_{rr} r^2 dr}. \quad (46)$$

where $f' = df/dr$.

V. CONCLUSIONS

In this paper we have derived a general criterion for the dynamical stability of anisotropic systems, by following the approach proposed by Dev & Gleiser. Starting from the pulsation equation and applying the conditions required to achieve the stability, we arrived to Eqs.(27)-(28).

We have subsequently applied the criterion to several models, by finding very interesting results. Starting from the anisotropic fermionic configurations studied in Paper I, we have found two limiting values for γ_{cr} , according to the level of anisotropy. When there is the equivalence between P_{rr} and P_t , we have recovered the well known relation $\gamma_{cr} = 4/3$. Amazingly, also in the full anisotropic regime, we have found a limiting value, i.e. $\gamma_{cr} = 14/9$.

The behavior of γ_{cr} (see also Fig. 1) shows clearly that an increase of the level of anisotropy yields harder the achievement of the stability. It seems that anisotropic fermionic configurations can unlikely exist, unless an external process intervene.

It should be stressed that Eq.(34) does not explicitly depend on the statistics considered, by implying that also configurations formed by boson or classical particles could present the same features. In particular, by considering classical systems, an interesting application of Eq.(34) could concern globular clusters or galactic halos. By now considering the other applications, we see that anisotropy can also makes easier the achievement of stability. If we now refer to Figs. 2 and 3, we see clearly that the stability can be achieved also for $\gamma < 4/3$. This means that configurations that can not exist in the isotropic regime could, in principle, exist when there is prevalence of tangential or radial pressure. These models can be use-

ful to analyze the stability of superdense neutron stars or super Chandrasekhar white dwarfs (Ref. [20] and references therein).

Surprisingly, Fig. 3 shows a second interesting case. In the limit $C \rightarrow 2\pi/3$, we have in fact that $\gamma_{cr} \rightarrow \infty$, by indicating that configurations with a high level of anisotropy can not exist. Similarly, also the stability analysis of the generalized polytropes presents the same features.

The cases listed in Sec. IV constitute only a small part of the wide range of applications that Eqs.(27)-(28) can have. However, it is important to stress the validity of Eq.(28) in presence of intense gravitational fields. The extension of the stability criterion to the framework of General Relativity is therefore a logical consequence. This problem will be addressed to a forthcoming publication.

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